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INSTABILITY OF UNIFORM FLOW

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SUMMARY

We consider uniform flow of a Newtonian fluid transverse to a domain bounded by parallel planes. We investigate the possibility of introducing instabilities in this flow by the choice of inflow and outflow conditions. Some instabilities of this kind are found.

KEY WORDS Open boundary conditions Flow stability

INTRODUCTION

Problems in computational fluid mechanics frequently involve 'open' boundaries (i.e. boundaries traversed by the fluid), which are introduced by truncation of the flow domain. In order to solve the equations governing the flow, it is necessary to impose boundary conditions at these open boundaries. These boundary conditions are not provided by the physics of the problem but are a mathematical artefact. Many possible choices of inflow and outflow conditions exist. A systematic mathematical theory of flow problems with open boundaries has yet to be developed. In this paper we investigate one aspect of the problem, namely the possibility of introducing artificial flow instabilities by the choice of open boundary conditions. We shall consider a flow which clearly has no instabilities in the absence of open boundaries, namely uniform flow. The inflow and outflow boundaries are given by two parallel planes which are transverse to the direction of the flow. While this problem is not too typical of applications, it has the essential feature of open boundaries and is easily accessible to analysis. We shall consider the following types of boundary conditions at the open boundaries:

(1) 'natural' boundary conditions
(2) vorticity and pressure conditions
(3) traction conditions upstream and velocity conditions downstream
(4) Dirichlet conditions upstream and 'absorbing' conditions downstream.

The first three cases will be seen to exhibit instabilities.

The flow domain is the strip $(0, 1) \times \mathbb{R}$ and we rescale all the variables in the Navier–Stokes equations so that all constants can be set to unity. Thus the governing equations are

$$\dot{\mathbf{u}} = \operatorname{div}[\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}} - p\mathbf{I} - \mathbf{u}\mathbf{u}^{\mathrm{T}}], \qquad \operatorname{div}\mathbf{u} = 0. \tag{1}$$

We have written the equations of motion in their natural conservation form, since we shall consider boundary conditions which are 'natural' for this form of the equation. We are interested in stability of a uniform flow in the x-direction, i.e. $\mathbf{u} = (U, 0)$, $p = 0$. When (1) is linearized for this flow, the following equations result:

$$u_t + Uu_x = u_{xx} + u_{yy} - p_x, \qquad v_t + Uv_x = v_{xx} + v_{yy} - p_y, \qquad u_x + v_y = 0. \tag{2}$$

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Here u and v denote the components of the perturbation to the velocity.

To complete the problem, one needs boundary conditions at the inflow boundary $x = 0$ and the outflow boundary $x = 1$. If conditions of prescribed velocity are imposed at both boundaries, then it is very easy to show by energy estimates that the uniform flow is stable. However, there are many other choices for which energy estimates do not seem to be useful in determining the stability.

NATURAL BOUNDARY CONDITIONS

The first possibility which we shall consider is natural boundary conditions. For equations (1) these are $2u_x - p - u^2 = v_x + u_y - uv = 0$. For the linearized problem this reduces to

$$2u_x - p - 2Uu = 0, \qquad v_x + u_y - Uv = 0. \tag{3}$$

We can simplify our equations by introducing a streamfunction: $u = -\psi_y$, $v = \psi_x$. This transforms (2) to the equation

$$\Delta\psi_t + U\Delta\psi_x = \Delta\Delta\psi. \tag{4}$$

After some calculation we find that the boundary conditions are transformed to

$$-\psi_{xxx} - 3\psi_{xyy} + \psi_{xt} + U\psi_{xx} + 2U\psi_{yy} = 0, \qquad \psi_{xx} - \psi_{yy} - U\psi_x = 0. \tag{5}$$

Consider now the ansatz $\psi(x, y) = \exp(\alpha x + i\alpha y)$. Being harmonic, this function satisfies (4) and a simple calculation shows that (5) is also satisfied if $\alpha = U/2$. Hence there is a neutrally stable mode at this wave number.

For general α equation (4) can be satisfied by the ansatz

$$\begin{aligned}\psi(x, y, t) = {} & \psi_1 \exp(\alpha x + i\alpha y + \lambda t) + \psi_2 \exp(-\alpha x + i\alpha y + \lambda t) \\ & + \psi_3 \exp(\beta_1 x + i\alpha y + \lambda t) + \psi_4 \exp(\beta_2 x + i\alpha y + \lambda t),\end{aligned} \tag{6}$$

where

$$\beta_{1,2} = \frac{U \pm \sqrt{[U^2 + 4(\alpha^2 + \lambda)]}}{2}. \tag{7}$$

By inserting (6) into the boundary conditions (5), we find that the determinant of the following 4×4 matrix must be zero:

$$\begin{aligned} a_{1i} &= \gamma_i^2 + \alpha^2 - U\gamma_i, \\ a_{2i} &= (\gamma_i^2 + \alpha^2 - U\gamma_i)\exp(\gamma_i), \\ a_{3i} &= -\gamma_i^3 + 3\alpha^2\gamma_i + \lambda\gamma_i + U\gamma_i^2 - 2U\alpha^2, \\ a_{4i} &= (-\gamma_i^3 + 3\alpha^2\gamma_i + \lambda\gamma_i + U\gamma_i^2 - 2U\alpha^2)\exp(\gamma_i). \end{aligned} \tag{8}$$

Here

$$\gamma_1 = \alpha, \qquad \gamma_2 = -\alpha, \qquad \gamma_3 = \beta_1, \qquad \gamma_4 = \beta_2. \tag{9}$$

We can now proceed numerically as follows. We fix $\lambda \geqslant 0$ and α and then vary U from zero until the determinant of the above matrix changes sign. Where it does, λ is an eigenvalue. The calculations verify that for $\lambda = 0$ the sign change occurs at $U = 2\alpha$. If we pick λ slightly positive, the sign change is shifted to a slightly larger value of U. This verifies that the flow is unstable to wave numbers less than $U/2$. We note that although the neutrally stable mode for $\alpha = U/2$ is irrotational ($\Delta\psi = 0$), the unstable modes for $\alpha < U/2$ are not irrotational.

The boundary conditions studied above seem to have been used only sporadically in numerical calculations, even though they are the natural boundary conditions when the Navier–Stokes equations are written in conservation form. Indeed, some other 'natural' conditions are used more commonly. One choice is to set the tractions equal to zero and omit the momentum transport terms in (3):

$$2u_x - p = 0, \qquad v_x + u_y = 0. \tag{10}$$

A numerical calculation analogous to that described above showed no instabilities for this case. Another common choice in numerical simulations is

$$u_x - p = 0, \qquad v_x = 0, \tag{11}$$

which is mathematically 'natural' for $\Delta\mathbf{u} - \nabla p$. Again a numerical calculation as above shows no evidence of instability. The procedure of looking for sign changes of the determinant would only detect instabilities from real eigenvalues. We also computed some complex eigenvalues using Newton's method and found only eigenvalues with negative real parts. It appears therefore that the boundary conditions (10) or (11) do not cause instability. It is interesting that the conditions which are 'natural' from the physical point of view lead to instabilities while more naive choices do not.

VORTICITY AND PRESSURE CONDITIONS

As our next example let us consider conditions of prescribed vorticity and pressure. In this case we need to consider (4) with boundary conditions

$$\psi_{xx} + \psi_{yy} = -\psi_{xxx} - \psi_{xyy} + \psi_{xt} + U\psi_{xx} = 0. \tag{12}$$

We can find the explicit solution

$$\psi = \exp(-\alpha x + i\alpha y + U\alpha t), \tag{13}$$

which clearly grows exponentially with time. The instability is even worse than the one above, since the growth rate tends to infinity as $\alpha \to \infty$, i.e. the problem is actually ill-posed. If we change the condition of prescribed pressure to prescribed normal traction, $2u_x - p = 0$, then our second boundary condition becomes

$$-\psi_{xxx} - 3\psi_{xyy} + \psi_{xt} + U\psi_{xx} = 0 \tag{14}$$

and our explicit solution is changed to

$$\psi = \exp(-\alpha x + i\alpha y + U\alpha t - 2\alpha^2 t). \tag{15}$$

Now we no longer have unlimited growth for large α but we still have instability for $\alpha < U/2$.

TRACTION UPSTREAM AND VELOCITY DOWNSTREAM

Let us consider (4) with Dirichlet conditions downstream:

$$\psi = \psi_x = 0 \quad \text{at } x = 1. \tag{16}$$

At the upstream boundary we prescribe the following boundary conditions:

$$-\psi_{xxx} - 3\psi_{xyy} + \psi_{xt} + U\psi_{xx} = 0, \qquad \psi_{xx} - \psi_{yy} + \varepsilon\psi_x = 0. \tag{17}$$

For $\varepsilon = 0$ this gives us traction boundary conditions upstream and velocity conditions downstream. Such a choice of boundary conditions is generally regarded as 'bad' by practitioners of computational fluid mechanics. Indeed, we shall see that for any positive ε, however small, we get instability if U is large enough. The critical U tends to infinity as $\varepsilon \to 0$, but only logarithmically. Thus even though the case $\varepsilon = 0$ appears to be linearly stable, it should be viewed as a marginal case which can easily be pushed over the edge. The situation is reminiscent of the well-known case of plane Couette flow, which is known to be linearly stable at all Reynolds numbers but is not observed to be stable in practice.

We specifically study the case where ψ does not depend on y, i.e. the perturbation to the flow is purely in the y-direction. Then (4) and the first equation of (17) lead to

$$\psi_{xt} + U\psi_{xx} = \psi_{xxx}, \tag{18}$$

which needs to be solved together with the boundary conditions (16) and the second condition in (17). For neutral stability we again look for time-independent solutions, $\psi_t = 0$. In that case (18) is solved by $\psi_x = c_1 + c_2 \exp(Ux)$ and by inserting into the boundary conditions, we find the condition

$$\varepsilon \exp(U) = U + \varepsilon, \tag{19}$$

i.e.

$$\varepsilon = \frac{U}{\exp(U) - 1}. \tag{20}$$

That is, if we choose a positive $\varepsilon > 0$, then instability occurs for large enough U, with a critical value which grows logarithmically as ε approaches zero. In contrast, we note that for negative U (i.e. velocity conditions upstream and traction conditions downstream) this instability will not occur if $\varepsilon < 1$.

Even if we set $\varepsilon = 0$, the flow is only theoretically stable. In (18) let $\psi_x = \gamma$, so that our equation becomes

$$\gamma_t + U\gamma_x = \gamma_{xx}, \tag{21}$$

and for $\varepsilon = 0$ we have the boundary conditions

$$\gamma_x(0) = \gamma(1) = 0. \tag{22}$$

It can be shown that this problem is stable, but the least stable eigenvalue behaves like $U^2 \exp(-U)$ for large U. Thus the decay rate becomes extremely small even for moderate values of U. It will hence take an extremely long time before solutions decay, and the decay rate tells us nothing about the behaviour at times of order one. Indeed, it is not hard to see how solutions will behave when U is large. If we ignore the term γ_{xx} in (21) and the downstream boundary condition, then the solution is given by

$$\gamma(x, t) = \begin{cases} \gamma_0(x - Ut) & \text{if } x > Ut, \\ \gamma_0(0) & \text{if } x < Ut. \end{cases} \tag{23}$$

Here γ_0 is the initial condition at $t = 0$. If U is large, (23) will be the dominant contribution to the solution, except for a boundary layer downstream which is needed to accommodate the boundary condition there. Hence a large gradient will form near the downstream boundary. The time it takes for this large gradient to form is only of order $1/U$, while the time it takes for the solution to decay to zero is of order $\exp(U)/U^2$.

Note that there is no instability or near instability if velocity conditions are prescribed upstream and tractions downstream. To see this, we simply multiply equations (2) by the velocity and integrate over the flow domain. After an integration by parts this yields the energy estimate

$$\frac{\partial}{\partial t}\int_\Omega |\mathbf{u}|^2 \,\mathrm{d}V = -U\int_{\Gamma_o} |\mathbf{u}|^2 \,\mathrm{d}S - \int_\Omega |\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}|^2 \,\mathrm{d}V, \tag{24}$$

where Ω is the flow region and Γ_o is the downstream boundary. From this, stability is immediate. In the situation considered above, i.e. velocities downstream and tractions upstream, the first term on the right of (24) is replaced by plus the integral over the upstream boundary and hence the energy equation allows no conclusion about stability.

ABSORBING CONDITIONS DOWNSTREAM

Finally we consider Dirichlet conditions upstream,

$$u = v = 0, \tag{25}$$

and the following conditions downstream:

$$u_t + Uu_x = v_t + Uv_x = 0. \tag{26}$$

If we think of waves travelling with speed U at high Reynolds number, then this means that the outflow boundary is absorbing these waves. We thus expect a stabilizing influence and we shall demonstrate that the flow is indeed stable. We write the Navier–Stokes equations in the form

$$\mathbf{u}_t + U\mathbf{u}_x = \Delta\mathbf{u} - \nabla p, \tag{27}$$

multiply by $\mathbf{u}_t + U\mathbf{u}_x$ and integrate. We find

$$\begin{aligned}\int_\Omega (\mathbf{u}_t + U\mathbf{u}_x)^2 \,\mathrm{d}V &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\mathbf{u}|^2 \,\mathrm{d}V - \frac{1}{2}\int_\Omega U\frac{\partial}{\partial x}|\nabla\mathbf{u}|^2 \,\mathrm{d}V - U\int_{\Gamma_i} v_x^2 \,\mathrm{d}S \\ &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\mathbf{u}|^2 \,\mathrm{d}V - \frac{1}{2}U\int_{\Gamma_o} |\nabla\mathbf{u}|^2 \,\mathrm{d}S - \frac{1}{2}U\int_{\Gamma_i} v_x^2 \,\mathrm{d}S.\end{aligned} \tag{28}$$

Here Γ_i and Γ_o are the inflow and outflow boundaries respectively. We conclude that $\int_\Omega |\nabla\mathbf{u}|^2 \,\mathrm{d}V$ decreases monotonically in time and the flow is stable.

CONCLUSIONS

As we have seen in the examples above, the possibility exists that the choice of boundary conditions at open boundaries can lead to artificial flow instabilities. One may expect similar phenomena to occur in more complicated flow situations which are more typical of applications. The effect on flow stability ought to be an important factor in a rational approach towards characterizing 'good' versus 'bad' choices for conditions at open boundaries.

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